Solutions to Games, Transitions and Efficiency

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The Phenomena

A solution concept, such as Nash equilibrium

- Strong belief assumptions
- Non simultaneous change (democracy, marriage, traffic)
- Lack of coordination

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How?

No theoretical modelling of using various solutions simultaneously

⇒ We

1. formally model a transition
2. bound efficiency

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Given

1. A game $G = (N, S = S_1 \times S_2 \times \ldots \times S_n, (u_i)_{i=1,\ldots,n})$
2. A solution concept (e.g., NE) defines a solution set $D \subseteq S$

To model movement or lack of coordination,

Definition

Given $D \subseteq S$, define a transition as any profile $s = (s_1, \ldots, s_n) \in S$ such that for each $i \in N$, there exists a solution $d(s, i) = (d_1, \ldots, d_n) \in D$, such that $s_i = d_i$.

Denote the set of all the transitions to be $T(D) \subseteq S$, the transition set.

By definition, $D \subseteq T(D)$ and $T(T(D)) = T(D)$
Classically,

- \( SW(s) \triangleq \sum_{i \in N} u_i(s) \)
- \( \text{PoA} \triangleq \frac{\min_{s \in D} SW(s)}{\max_{s \in S} SW(s)} \) and \( \text{PoS} \triangleq \frac{\max_{s \in D} SW(s)}{\max_{s \in S} SW(s)} \)

Given

1. \( G = (N, S, (u_i)_{i=1,...,n}) \)
2. \( D \subseteq S \)

We define

- \( \text{PoTA} \triangleq \frac{\min_{s \in T(D)} SW(s)}{\max_{s \in S} SW(s)} \) and \( \text{PoTS} \triangleq \frac{\max_{s \in T(D)} SW(s)}{\max_{s \in S} SW(s)} \)
General Bounds

Always holds

\[ \text{PoTA} \leq \text{PoA}, \text{PoTS} \geq \text{PoS}, \]

but not the other direction, generally speaking.
General Bounds - Individual Utilities

**Definition**

*Player i’s utility over profile set* $A \subseteq S$ *is* $\alpha$-*lower (-upper) dependent on coordination* if

$$\min_{s \in T(A)} u_i(s) \geq \min_{t \in A} u_i(t)/\alpha \quad \text{and} \quad \max_{s \in T(A)} u_i(s) \leq \alpha \cdot \max_{t \in A} u_i(t).$$

**Definition**

The utility of agent $i$ is $\beta$ *varied over* $A \subseteq S$ *if for all profiles* $s, t$ *in* $A$,

$$SW(s) \geq SW(t) \Rightarrow u_i(s) \geq u_i(t)/\beta.$$

For example, the utility of a game with identical payoff functions is $1$-upper dependent on coordination and $1$ varied over any set.
Consider a game $G = (N, S, (u_i)_{i=1,\ldots,n})$ with a solution set $D \subseteq S$, such that over $D$, the utility of every player $i$ is $\beta$ varied and $\alpha$-lower dependent on coordination, then

$$\text{PoTA} \geq \frac{\text{PoA}}{\alpha \beta}.$$  \hfill (1)

If for every player $i$, its utility over $D$ is $\beta$ varied and $\alpha$-upper dependent on coordination, then

$$\text{PoTS} \leq \alpha \beta \text{ PoS}.$$  \hfill (2)

For example, an identical utility game has $\text{PoTS} = \text{PoS}$
Now, concentrate on $NE$ and $T(NE)$

**Proposition**

In a two-player game, if for every $x, x' \in S_1$ and every $y, y' \in S_2$ there holds the implication

\[
\begin{align*}
&u_1(x, y) \leq u_1(x', y) \text{ and } u_2(x, y) \leq u_2(x, y') \\
\Rightarrow & SW(x, y) \leq SW(x', y) \text{ or } SW(x, y) \leq SW(x, y'),
\end{align*}
\]

then we have $\text{PoTS} = \text{PoS}$.

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A game is $\alpha, \beta, \lambda, \mu$-extensively smooth if the following holds:

1. $\forall s^* \in \arg\max\{SW(s) : s \in S\}$, $\forall t \in T(NE)$:
   $\sum_{i=1}^{n} u_i(s_i^*, t_i - i) \geq \lambda SW(s^*) - \mu SW(t)$.

2. $\forall i \in N$, $\forall s \in T(NE)$, $\forall d \in NE$ such that $s_i = d_i$:
   $u_i(s) \geq \alpha u_i(d)$.

3. $\forall s^* \in \arg\max\{SW(s) : s \in S\}$, $\forall t, v \in T(NE)$:
   $u_i(s_i^*, t_i - i) \geq \beta u_i(s_i^*, v_i - i)$.

Proposition
Any $\alpha, \beta, \lambda, \mu$-extensively smooth game has $\text{PoTA} \geq \alpha \beta \lambda \frac{1}{1 + \alpha \beta \mu}$.
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**Proposition**

Any $\alpha, \beta, \lambda, \mu$-extensively smooth game has $\text{PoTA} \geq \frac{\alpha \beta \lambda}{1 + \alpha \beta \mu}$.

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Proposition

For an identical utility game, the PoS = PoTS = 1, but the price of anarchy can be arbitrarily low.

If we also have that the best response strategies of any player $i$ to the strategies $s_{-i}$ of the others do not depend on those $s_{-i}$, then PoTA = PoA = PoS = PoTS = 1.

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Non-Atomic Routing Game Bound - Definitions

**Definition**

1. **Source and sink pairs** \((s_1, t_1), \ldots, (s_k, t_k)\)
2. **Each commodity is of size** \(r_i\) **to be routed through paths in** \(\mathcal{P}_i\)
3. **A flow vector** \(f \in \mathbb{R}^{|\mathcal{P}|}\) **is feasible** if \(\sum_{P \in \mathcal{P}_i} f_P = r_i\)
4. **Each edge has a non-decreasing cost function** \(c_e : \mathbb{R}_+ \rightarrow \mathbb{R}_+\)
5. **Define** \(c_P(f) \triangleq \sum_{e \in P} c_e(f_e)\)
6. **Define an equilibrium flow** as a feasible flow \(f\) such that for every commodity \(i = 1, \ldots, k\), for every path \(P \in \mathcal{P}_i\) such that \(f_P > 0\) and for every path \(P' \in \mathcal{P}_i\), we have \(c_P(f) \leq c_{P'}(f)\)
Definition

1. Define \( c_P(f) \overset{\Delta}{=} \sum_{e \in P} c_e(f_e) \)
2. An equilibrium flow

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Non-Atomic Routing Game Bound - Definitions Cont.

Definition

1. Define \( c_P(f) \triangleq \sum_{e \in P} c_e(f_e) \)
2. An equilibrium flow
3. \( C(f) \triangleq \sum_{P \in \mathcal{P}} c_P(f) \cdot f_P \)
4. Define the PoA as \( \frac{\text{the cost of the equilibrium flow}}{\text{the optimum cost}} \)
5. Define a transition as a feasible flow such that \( f_P > 0 \implies \) there exists an equilibrium flow \( f' \) with \( f'_P > 0 \)
6. Define the PoTA (PoTS) as \( \frac{\text{the cost of a most costly (cheapest) transition}}{\text{the optimum cost}} \)

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Example

- The only commodity with $r = 1$
- One equilibrium and a continuum of transitions
- PoA = PoS = PoTS = 1
- However, PoTA = $n$

Figure: Having $n$ parallel edges with $c_e(x) = x$ each.
Theorem

Given a set of cost functions $C$, a routing game and a commodity $i$, define

$$S_i(C) \triangleq \frac{\max \{|P| : P \in \mathcal{P}_i\} \sup_{c \in C} (c(r_i + \sum_{j \in \{1, \ldots, k\} \setminus \{i\}} r_j))}{\min \{|P| : P \in \mathcal{P}_i\} \inf_{c \in C} c(r_i/|\mathcal{P}_i|)}.$$  \hspace{1cm} (4)

Then, $\text{PoTA} \leq \text{PoA} \cdot \max_{i=1, \ldots, k} S_i(C)$, and this bound is tight.
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Then, $\text{PoTA} \leq \text{PoA} \cdot \max_{i=1,\ldots,k} S_i(C)$, and this bound is tight.

In particular, if $c_e(x) = a_e \cdot x$, such that $a_{\text{min}} \leq a_e \leq a_{\text{max}}$ and also the paths of different commodities never intersect, then

$$S_i(C) = \frac{\max \{|P| : P \in \mathcal{P}_i\} a_{\text{max}}}{\min \{|P| : P \in \mathcal{P}_i\} a_{\text{min}}} |\mathcal{P}_i|.$$  \hspace{1cm} (5)
Conclusions

1. Modelling lack of coordination
2. General efficiency bounds are appalling $\Rightarrow$ coordinate
3. Most NE bounds are not promising $\Rightarrow$ coordinate
4. The bounds are optimistic for
   - identical utility game with independent best responses
   - routing games with linear costs, non-intersecting commodities, similar path lengths per commodity, close cost functions, and few paths per commodity

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Future Work

- Limited transitions
- Repeated game
- Combining solutions from different solution concepts
Thank You!

Questions?